On ω—incompleteness of an Axiomatic Number Theory※

By

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1. The relativization of quantifiers is useful in proving the undecidability of formal theories.[1)] In the present paper a complicated relativization is considered, by which an independence in an axiomatic number theory is proved; i. e., a problem in 〔1〕[2)] (p. 279, footnote) is solved.

The relativization applies to a particular number theory Σ, which is formalized within the first order predicate logic (with the usual primitive symbols), and has the following axioms for number theory. Axioms for number theory.

(J_1) $\quad (x)(x=x)$,

(J_2) $\quad (x)(y)(x=y\supset(A(x)\supset A(y)))$,

$(<_1)$ $\quad (x)(\neg.\ x<x)$

$(<_2)$ $\quad (x)(y)(z)(x<y\ \&\ y<z\supset x>z)$,

$(<_3)$ $\quad (x)(x>x')$.

Rules of inference of Σ are as follows;

$$R_1\ \frac{A\quad A\supset B}{B}\qquad R_2\ \frac{C\supset A(x)}{C\supset(x)A(x)}\qquad R_3\ \frac{A(x)\supset C}{(\exists x)A(x)\supset C}$$

$$R_4\ \frac{A(0)\qquad (x)(A(x)\supset A(x'))}{A(a)},$$

where C does not contain x free, a is an arbitrary term, and in R_4, x is not contained in A(a).

Thus, in terms of the above stipulation the problem to be solved here is, "Is$(<_4)$ independent of Σ or not? "

$(<_4)$ $\quad (x)(y)(x<y\supset x'=y\vee x'<y)$.

The proof of the independence proceeds in the following manner.

A is valid in Σ, if and only if the correlated formula A is valid in the correlated theory $\Sigma^{(U)}$. Every formula valid in $\Sigma^{(U)}$ is "satisfiable". On the other hand, the formula $(<_4)^{(V)}$ (correlated with $(<_4)$) is not "satisfiable". Therefore $(<_4)$ is independent of Σ.

※ Contribution from the Shimonoseki College of Fisheries No. 282,
Received Aug. 5, 1959.

$A^{(U)}$ and $\Sigma^{(U)}$ are defined in section 2. Section 3 deals with the definition of the notion "satisfiability", and the proof of the independence of ($<_4$) therewith. In section 4, several consequences from the independence are pointed out.

2. Relativization.

2.1. Relativization of quantifiers.

In this section $U(\cdots,\cdots)$ is a definite binary predicate not contained in Σ.

Given any formula F and a variable t, which is not contained in F, then F° is obtained by replacing every sub-formula of the form $(x)\,G(x)$ or $(\exists x)\,G(x)$, with a formula $(x)(U(t,x) \supset G(x))$ or $(\exists x)(U(t,x)\ \&\ G(x))$, respectively. t is called the sub-variable of F. Moreover, if F contains free variables $a_1, \ldots, a_n$, then F^* is introduced with

$$(U)t,\ a_1) \supset (U(t,\ a_2) \supset (\cdots\cdots(U(t,\ a_n) \supset F^\circ))\cdots),$$

where t is the sub-variable of F. (If F contains no free variable, F^* is F°.) Thus, in F all variables are correlated with $U(\cdots,\cdots)$. Let $(\exists u)(t)\,U(u,t) \supset F)$ be denoted by $F^{(U)}$, provided that u is a variable not contained in F.

As the set of all free variables is linearly ordered by a fixed ordering, $F^{(U)}$ is uniquely determined.

The transformation from F to $F^{(U)}$ is called the relativization of quantifiers, and F is called relativized to $F^{(U)}$ for $U(\cdots,\cdots)$.

2.2. Relativized theory.

The relativized theory $\Sigma^{(V)}$ from Σ for $U(\cdots,\cdots)$ is obtained by the relativization of quantifiers of all formulas in Σ, as shown below;

2.2.1. Primitive symbols in $\Sigma^{(U)}$ are those of Σ or $U(\cdots,\cdots)$.

2.2.2. All the formulas of $\Sigma^{(U)}$ are of the form $F^{(U)}$ correlated with a formula F of Σ.

2.2.3. $A^{(U)}$ is an axiom of $\Sigma^{(U)}$ if and only if A is an axiom of Σ.

Rules of inference of $\Sigma^{(U)}$ are as follows;

$$R_1{}^{(U)}\quad \frac{A^{(U)}\quad (A \supset B)^{(U)}}{B^{(V)}} \qquad\qquad R_2{}^{(U)}\quad \frac{(C \supset A(x))^{(U)}}{(C \supset (x)\,A(x))^{(U)}}$$

$$R_3{}^{(U)}\quad \frac{(A(x) \supset C)^{(U)}}{((\exists x)A(x) \supset C)^{(U)}} \qquad\qquad R_4{}^{(U)}\quad \frac{(A(0))^{(U)}\quad ((x)(A(x) \supset A(x')))^{(U)}}{(A(a))^{(U)}},$$

where the conditicns for free variable x and free term a are the same as in Σ. Since the relativization is isomorphic, if A is a valid formula in Σ, then $A^{(U)}$ is derivable in $\Sigma^{(U)}$, and vice versa. Thus, if Σ is consistent, then $\Sigma^{(U)}$ is consistent.

3. An interpretation of the relativization.

In the following, a notion "satisfiability" is defined such that all the relativized axioms are satisfiable, and that the relativized rules of inference in $\Sigma^{(V)}$ conclude from satisfiable formulas to a satisfiable formula. Hence, if a formula F is provable in Σ, then $F^{(U)}$ is satisfiable. Finally, only if $(<_4)^{(U)}$ is proved to be unsatisfiable, then the problem dealt in this paper is positively solved.

3.1. Domain $\triangle$.

$\triangle$ is obtained from two individual constants o and w, by applying operation $'$ finite times, between two elements of which the following conditions hold; [3)]

3.1.1. For arbitrary elements a, b, c, $a=a$ and $a<a'$ are true, and $a<a$ is false. If $a<b$ and $b<c$ are true, then $a<c$ is true.

3.1.2. For any natural numbers m and n,

$$O^{(n)}<O^{(m)},\ w^{(n)}<O^{(m)},\ O^{(n)}<w^{(m)}, \text{ and } w^{(n)}<w^{(m)}$$

are equivalent to each other.

3.1.3. For any natural number m and n,

$$O^{(n)}=w^{(m)} \text{ and } w^{(n)}=O^{(m)} \text{ are false,}$$

where by $a^{(n)}$ we understand n-times application of operation $'$ on a.

3.2. $U(\cdots,\cdots)$.

Predicate $U(a,b)$ between a and b, which are elements of $\triangle$, is specified as follows;

3.2.1. For any x of $\triangle$ such that $x=w^{(n)}$, $U(x, O)$ and $U(x, x)$ are true.

3.2.2. If, for any x and y of $\triangle$, $U(x, y)$ is true, then $U(x, y')$ and $U(x', y')$ are true.

3.2.3. If there exists y, an element of $\triangle$ such that $U(x, y)$ is true, for any natural number n, $x=O^{(n)}$ is false.

3.3. Satisfiability.

The notion "satisfiability" is defined in the well-known way. All variables are assumed to be linearly ordered, and let the i-th variable be denoted by x_i. "Satisfiability" is defined for all the formulas, which are constructed from two individual constants o and w by the usual rules of formation.

Let the set of all infinite sequences of elements of $\triangle$ be denoted by S, and the i-th term of $f\leftarrow s$ be denoted by $f(i)$.

3.3.1. T.

A function T on formulas is recursively defined as follows;

3.3.1.1. $T(x_i=x_j)$ is the set of all $f\leftarrow S$ such that $f(i)=f(j)$.

3.3.1.2. $T(x_i=a^{(n)})$ (a is O or w) and $T(a^{(n)}=x_i)$ are the set of all $f\leftarrow S$ such that $f(i)=a^{(n)}$.

3.3.1.3. For any n, $T(a^{(n)}=a^{(n)})\equiv S$. ("$\equiv$" means the extensional identity in the set theory.)

3.3.1.4. For $m\neq n$, $T(a^{(m)}=a^{(n)})\equiv\theta$,

where θ is the empty set.

3.3.1.5. For any m and m, $T(O^{(m)}=w^{(n)})\equiv T(w^{(m)}=O^{(n)})\equiv\theta$.

3.3.1.6. $T(x_i<x_j)$ is the set of all $f\rightarrow S$ such that $f(i)<f(j)$.

3.3.1.7. $T(a^{(n)}<x_i)$ is the set of all $f\rightarrow S$ such that $a^{(n)}<f(i)$.

3.3.1.8. $T(x_i<a^{(n)})$ is the set of all $f\leftarrow S$ such that $f(i)<a^{(n)}$.

3.3.1.9. $T(a^{(m)}<a^{(n)})\equiv S$, if $n>m$,

$T(a^{(m)}<a^{(n)})\equiv\theta$, if $n\leqq m$.

3.3.1.10. $T(O^{(m)} < w^{(n)}) \equiv T(w^{(m)} < O^{(n)}) \equiv T(w^{(m)} < w^{(n)}) \equiv T(O^{(m)} < O^{(n)})$.

3.3.1.11. $T(U(x_i, x_j))$ is the set of all $f \rightarrow S$ such that $U(f(i), f(j))$.

3.3.1.12. $T(U(a^{(n)}, x_i))$ is the set of all $f \rightarrow S$ such that $U(a^{(n)}, f(i))$.

3.3.1.13. $T(U(x_i, a^{(n)}))$ is the set of all $f \leftarrow S$ such that $U(f(i), a^{(n)})$.

3.3.1.14. $T(U(O^{(m)}, a^{(n)})) \equiv \theta$.

3.3.1.15. $T(U(w^{(m)}, a^{(n)}))$ is the set of all $f \rightarrow S$ such that $U(w^{(m)}, a^{(n)})$. Let A and B be any formulas.

3.3.1.16. $T(\neg A) . \equiv . S - T(A)$.

3.3.1.17. $T(A \vee B) . \equiv . T(A) + T(B)$.

3.3.1.18. $T(A \mathbin{\&} B) . \equiv . T(A)\ T(B)$.

3.3.1.19. $T(A \supset B) . \equiv . S - T(A) + T(B)$.

3.3.1.20. $T((x_i) A(x_i))$ is the set of all $f \rightarrow S$ such that any $g \leftarrow S$ which differs from at most the i-th place belongs to $T(A(x_i))$.

3.3.1.21. $T((\exists x) A(x_i)) . \equiv . T(\neg (x_i) \neg A(x_i)) . \equiv . S - T((x_i) \neg A(x_i))$.

3.3.2. A formula A is called satisfiable, if $T(A) . \equiv . S$.

3.4. The axioms of $\Sigma^{(V)}$ are satisfiable.

Proof. From the definitions 3.3.1.16—21, it is easily seen that,

3.4.1. Logical axioms are satisfiable.

3.4.2. From 3.3.1.1 $(J_1)^{(U)}$ is satisfiable.

Similarly, $(J_2)^{(U)}$, $(<_1)^{(U)}$, $(<_2)^{(U)}$, and $(<_3)^{(U)}$ are satisfiable.

3.4.3. R_1 infers satisfiable formulas to a satisfiable formula.

Assume that $A^{(U)}$ and $(A \supset B)^{(U)}$ are satisfiable.

Then, there exist natural numbers n and m such that

$(t)(U(w^{(m)}, t) \supset A^*)$ and $(t)(U(w^{(n)}, t) \supset (A \supset B)^*)$

are satisfiable, where $(A \supset B)^*$ is of the form

$(U(t, a_1) \supset (\cdots\cdots (U(t, a_k) \supset (A \supset B)^{\circ}) \cdots\cdots)$, with free variable $a_1, \cdots, a_k$ contained in $A \supset B$.

Let max (n, m) be l. Then

$(t)(U(w, t)\ A)$ and $(t)(U(w, t)(A\ B))$ are satisfiable.

Hence,

$(t)(U(w^{(l)}, t) \supset (U(t, a_1) \supset (\cdots (U(t, a_k) \supset A^{\circ} \mathbin{\&} (A \supset B)^{\circ}) \cdots)$ are satisfiable.

Since $(A \supset B)^{\circ}$ is $A^{\circ} \supset B^{\circ}$, $(t)(U(w^{(l)}, t) \supset (U(t, a_1) \supset (\cdots (U(t, a_k) \supset B^{\circ},) \cdots)$. is satisfiable. Thus, $B^{(U)}$ is satisfiable.

3.4.4. R_2 and R_3 infer from a satisfiable formula to another satisfiable one.

Proof is easy from the definition of T.

3.4.5. R_4 infers from a satisfiable formulas to another satisfiable one. Lemma 3.4.6. is useful in the proof of 3.4.5.

3.4.6. If, for any n, $(A(O^{(n)}))^{(U)}$ is satisfiable, then there exists a natural number N such that, for any $m > N$, $(A(w^{(n)}))^{(U)}$ is satisfiable.

Proof of 3.4.6. is made by the induction on the number of quantifiers in $A(o^{(n)})$.

3.4.6.1. The case in which $A(O^{(n)})$ has no quantifier.

Assume that $(A(O^{(n)}))^{(U)}$ is satisfiable, for any n where $A(O^{(n)})^{(U)}$ is of the form $(\exists u)(t)(U(u, t) \supset A(O^{(n)})^{*})$.

Then, there exists a natural number M, such that $(t)(U(\omega^{(M)}, t) \supset A(O^{(n)})^{*})$ is satisfiable. From the definitions of T and $U(\cdots,\cdots)$, for any $m > M$ $(t)(U(w^{(m)}, t) \supset A(O^{(n)})^{*})$ is satisfiable, especially, for N the number of all symbols in $A(O^{(n)})$, $(t)(U(w^{(N)}, t) \supset A(O^{(n)})^{*})$ is satisfiable. Then, $(t)(U(w^{(N)}, t) \supset A(w^{(n)})^{*})$ is satisfiable. Thus, there exists a natural number N such that, for any $n > N$, $A(w^{(n)})^{*}$ is satisfiable.

3.4.6.2. Suppose that if for any m, $B(O^{(m)})$, with at most k 1—1 quantifiers, is satisfiable, then there exists a natural number N such that, for any $n > N$, $B(w^{(n)})^{(V)}$ is satisfiable.

Let $((x) A(x, O^{(n)}))^{(U)}$, with (k+1)—quantifiers, be satisfiable for any n. Then $(A(x, O^{(n)}))^{(U)}$ is satisfiable for an arbitrary natural number n. Therefore, there exists N such that, for any $m > N$, $(A(x, w^{(m)}))^{(U)}$ is satisfiable since $A(x, w^{(m)})$ has only k—quantifiers. Hence, there exists N such that, for any $m > N$, $((x) A(x, \omega(m))^{(U)}$ is satisfiable.

3.4.6.3.[6] In the case in which A is of the form $B \vee D$, $B \& D$, $B \supset D$, and $\neg B$, it is easily shown that If, for B and D, 3.4.6. holds, then, for A, 3.4.6. holds.

3.4.6.4. The case in which A is of the form $(\exists x) D(x, O^{(n)})$ is easily treated by 3.4.6.2—3.

Thus, 3.4.6. is proved.

Proof of 3.4.5.

Assume that $(A(O))$ and $(x)(A(x) \supset A(x'))$ are satisfiable.

Then, from 3.4.3, for arbitrary natural number n, $A(O^{(n)})$ is satisfiable. Therefore, by 3.4.6. there exists N_0 such that, for any $m > N_0$, $(A(w^{(m)}))^{(U)}$ is satisfiable.

Hence, $(t)(U(w^{(N_0)}, t) \supset (U(t, O^{(n)}) \supset A(O^{(n)})^{*})$ and

$$(t)(U(w^{(N_0)}, t) \supset (U(t, w^{(n)}) \supset A(w^{(n)})^{*}) \text{ are satisfiable.}$$

Then, $(t)(U(w^{(N_0)}, t) \supset A(a)^{*})$ is satisfiable, where a is a free variable.

3.5. From the above proof any formula derivable in $\Sigma^{(V)}$ is satisfiable. Hence, if a formula F is provable in Σ, then $F^{(U)}$ is satisfiable. But $(<_4)^{(U)}$ is not satisfiable, for, for in any n,

$$O^{(n)} < w^{(n+1)} \supset O^{(n+1)} = w^{(n+1)} \vee O^{(n+1)} < w^{(n+1)}$$

is false. Therefore, $(<_4)$ is independent of Σ. Thus, the problem in [1] is solved.

4. Consequences.

From the result obtained in § 3, several consequences are obtained,

4.1. The following formulas are independent of.

$(a)(b)(A(b) \ \& \ (x)(A(x) \supset A(x')) \supset (a = b \vee b < a \supset A(a)))$

$(x)(y)(x < y \supset x' < y')$

$(x)(y)(x < y \vee x = y \vee y < x)$

It is easy to prove them.

4.2. Σ is ω − incomplete.

Proof. Denote ($<_4$) by (x) (y) A (x, y). A ($O^{(n)}$, $O^{(m)}$) is valid in Σ for any m and n. But (x) (y) (x, y) is independent.

References

1) TARSKI, Cf. A. 1953. Undecidable Theories, I. 5.

2) [1], HILBERT, D. u. P. BERNAYS, 1934. Grundlagen der Mathematik, vol. 1.

3) "=" and ">" moreover O in this section are new symbols different from those in §1, but no confusion is likely to occur.

4) For EXAMPLE, cf. S. C. KLEENE, 1952, Introduction to Metamathematics, §50.

5) For EXANPLE, cf. E. MENDELSON, (1956), Some Proofs of Independence in Axiomatic Set Theory, *Journal of Symbolic Logic*, vol. 21 p. 293—4.

6) Those are easily shown by the following proposition. Let A (a_1, ... a_ℓ, $o^{(n)}$ have no more individual constants than a_1, ... a_ℓ, If and only it A (a_1, ... a_ℓ, $o^{(n)}$) is satisfiable, A ($\tilde{a}_1$, ..., $\tilde{a}_\ell$ $a^{(n)}$) is satisfiable, where $\tilde{a}$ is $o^{(m)}$ if a is $\omega^{(n)}$, or $\tilde{a}$ is $\omega^{(m)}$ if a is $o^{(m)}$.