Formal logic and information modeling

Sohei Ito
National Fisheries University, Japan
About me

• Sohei Ito
  – Assistant prof. at Department of Fisheries Distribution and Management, National Fisheries University
    • Located in Shimonoseki, Japan
Shimonoseki

• Surrounded by sea
• Well-known as Fugu (a quality fish)
My Research

• Formal verification of software systems
  – Correct optimizer in compiler construction
  – Higher-order program verification

• Information modeling
  – Business process modeling
  – Biological system modeling

Common ground is logic!
Why do we use logic?

• Rigorous syntax and semantics which enable
  – Formal representation of reality or problems
  – Theoretical reasoning
  – Automated reasoning (for a certain class of logic)

Formal treatment of something!

• Software/Hardware systems
• Business processes
• Enterprises
• Biological systems
• Etc.
This talk

- **Introduction to logic**
  - First-order logic
  - Modal logic

- **Information modeling using logic**
  - Information vs. knowledge
  - Ontology

- **Example ontology**
  - Block world
Introduction to logic

• Logic is a language.
  – Artificial language.

• Originally invented to formalize mathematical notions and proofs.
  – To enhance precise understanding.
Components of language

- Syntax (grammar)
- Semantics (meaning)

Example of English.

- Syntax: Subject-verb-object.
  - I play football.
- Semantics:
  - Subject = a thing which is acting
  - Verb = action
  - Object = a thing being acted upon
Components of logic

• Syntax = what are sentences?
  – How to write terms & formulas

• Semantics = is it true/false?
  – How to interpret terms and formulas

• Reasoning rules = can we prove it?
  – How to derive conclusions from premises
This talk

• Introduction to logic
  – First-order logic
  – Modal logic

• Information modeling using logic
  – Information vs. knowledge
  – Ontology

• Example ontology
  – Block world
First-order logic

• The logic which discusses **objects and properties** (relations).
  – Example sentences
    • 2 is an even number.
    • All even numbers are a sum of two odd primes.
    • All apples are red.
    • There exists an apple which is not red.
Syntax of first-order logic

- Predicate symbols: $P, Q, R, ...$
- Function symbols: $f, g, h, ...$
- Constant symbols: $a, b, c, ...$
- Variables: $x, y, z, ...$
- Connectives: $\lor, \land, \rightarrow, \neg, \forall, \exists$

Each predicate symbol and function symbol have its **arity** (the number of arguments)
Terms of first-order logic

• Terms
  – Constant symbols and variables are terms.
  – If $t_1, \ldots, t_n$ are terms, $f(t_1, \ldots, t_n)$ is a term.
    • The arity of $f$ is $n$

• Example
  – $\text{plus}(x, y)$
  – $\text{plus}(1, \text{plus}(x, y))$
  – $\text{father}(John)$

$\text{plus}$ is a binary function symbol (i.e. arity=2)
Formulas of first-order logic

- Formulas
  - If $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a formula.
  - If $\varphi$ and $\psi$ are formulas, then $\varphi \land \psi$, $\varphi \lor \psi$ and $\varphi \rightarrow \psi$ are formulas.
  - If $\varphi$ is a formula, then $\neg \varphi$ is a formula.
  - If $\varphi$ is a formula, then $\forall x \varphi$ and $\exists x \varphi$ are formulas.

$n$ is the arity of $P$
Example formulas

• $\forall x \exists y (x = 2y)$
• $\forall x (apple(x) \rightarrow red(x))$
• $\exists x (apple(x) \land \neg red(x))$
• $father(John) = Bob$
• $\forall x \exists y (father(x) = y)$

*apple* and *red* are unary predicate symbols (i.e. arity=1)
Semantics of first-order logic

• To decide whether the formula holds or not.
• Ex) $\forall x \exists y (x = 2y)$
  – Is it true?
    • If the domain is natural numbers, it is false.
    • If the domain is rational numbers, it is true.
  – It depends on the domain of $x$ and $y$.
    • Moreover, it depends on how we interpret the relation “$=$“ and the function “$\times$“.
Semantic structure of first-order logic

• Structure $\langle D, R, F \rangle$
  – $D$: domain of discourse (the set of objects)
  – $R$: the set of relations
    • Relation $p \subseteq D \times \cdots \times D$
  – $F$: the set of functions
    • Function $f: D \times \cdots \times D \to D$
Examples of relations/functions

- The relation “less than(\(<\))“ on natural numbers.
  - \(1<2, 1<3, \ldots, 2<3, 2<4, \ldots, 3<4, 3<5, \ldots\)
  - \(\leq \{ (1, 2), (1, 3), \ldots, (2, 3), (2, 4), \ldots \}\)
  - \(\leq \subseteq N \times N\)

- The function “plus(\(\oplus\))” on natural numbers.
  - \(2+3\mapsto 5, 12+7\mapsto 19, \ldots\)
  - \(\oplus: N \times N \rightarrow N\)
Interpretation of symbols

• Interpretation $I$ under $\langle D, R, F \rangle$
  - $I(P) \in R$ for each predicate symbol $P$
  - $I(f) \in F$ for each function symbol $f$
  - $I(c) \in D$ for each constant symbol $c$
  - $I(x) \in D$ for each variable $x$
  - $I(f(t_1, ..., t_n)) = I(f)(I(t_1), ..., I(t_n))$
Example of interpretation

• Predicate symbol $E$.
• Function symbol $m$.
• Constant symbol $\bar{2}$.

$I_1 = \{E \leftrightarrow =, m \mapsto \times, \bar{2} \mapsto 2\}$

$I_2 = \{E \leftrightarrow =, m \mapsto -, \bar{2} \mapsto 2\}$

$\langle \langle \mathbb{Z}, \{=, \{\times, -, \}\}, I_1 \rangle, \models \neg \forall x \exists y \left( E(x, m(\bar{2}, y)) \right) \rangle$

$\langle \langle \mathbb{Z}, \{=, \{\times, -, \}\}, I_2 \rangle, \models \forall x \exists y \left( E(x, m(\bar{2}, y)) \right) \rangle$

Interpreted as $x = 2 \times y$

Interpreted as $x = 2 - y$
Semantics of formulas

Precisely, $\langle\langle D, R, F \rangle, I \rangle$

- $I \models P(t_1, ..., t_n)$ iff $\langle I(t_1), ..., I(t_n) \rangle \in I(P)$
- $I \models \neg \varphi$ iff not $I \models \varphi$
- $I \models \varphi \land \psi$ iff $I \models \varphi$ and $I \models \psi$
- $I \models \varphi \lor \psi$ iff $I \models \varphi$ or $I \models \psi$
- $I \models \varphi \rightarrow \psi$ iff not $I \models \varphi$ or $I \models \psi$
- $I \models \forall x \varphi$ iff $I[x \mapsto d] \models \varphi$ for all $d \in D$
- $I \models \exists x \varphi$ iff $I[x \mapsto d] \models \varphi$ for some $d \in D$

$I[x \mapsto d]$ equals $I$ except it maps $x$ to $d$
Example

• Predicate symbols: \( \text{son, grandson} \)
• \( D = \{\text{John, Bob, Dan}\} \)
• \( R = \{S_1, S_2\} \)
  – \( S_1 = \{(\text{John, Bob}), (\text{Bob, Dan})\} \)
  – \( S_2 = \{(\text{John, Dan})\} \)
• \( I = \{\text{son \mapsto S_1, grandson \mapsto S_2}\} \)

• \( I \models \text{son}(\text{John, Bob}) \)
• \( I \models \neg \text{son}(\text{John, Dan}) \)
• \( I \models \text{grandson}(\text{John, Dan}) \)
• \( I \models \forall x \forall y \forall z (\text{son}(x, y) \land \text{son}(y, z) \rightarrow \text{grandson}(x, z)) \)
Deductive system

• Logic has a deductive system (reasoning rules) to reason about correct facts (theorems).

• Example rules
  – Modus ponens (→-elimination)
    \[
    \begin{array}{c}
    \varphi \\
    \varphi \rightarrow \psi
    \end{array}
    \quad
    \begin{array}{c}
    \psi
    \end{array}
    \]
  – ∀-elimination
    \[
    \begin{array}{c}
    \forall x \varphi(x)
    \end{array}
    \quad
    \begin{array}{c}
    \varphi(t)
    \end{array}
    \]
This talk

- Introduction to logic
  - First-order logic
  - Modal logic
- Information modeling using logic
  - Information vs. knowledge
  - Ontology
- Example
  - Formal REA model and ontology
Modal logic

• A logic that a truth of sentences can vary.
  – According to configurations of a piece of reality (state of affairs).

• Example
  – “It’s sunny.”
    • The interpretation of this sentence differs from day to day.
Formulas of modal logic

- Formulas in first-order logic are formulas.
- If $\varphi$ is a formula, $\Box \varphi$ and $\Diamond \varphi$ are formulas.
Semantics of modal logic (informal)

• □φ means “φ is necessary.”
• ◊ φ means “φ is possible.”

• How do we interpret “necessary” and “possible”?
Semantic structure of modal logic

- Frame $\langle W, A, D, R, F \rangle$
  - $W$: set of possible worlds
  - $A$: accessibility relation over $W$
  - $D$: domain of discourse
  - $R$: set of intensional relations
  - $F$: set of intensional functions
Possible world and accessibility

• Each world represents a state of affair.
• Accessibility relation represents accessible worlds from each world.
  – It represents how the world changes.

\[ W = \{ w_1, w_2, w_3, w_4 \} \]
\[ A = \{ (w_1, w_2), (w_1, w_4), (w_3, w_3), (w_4, w_1), (w_4, w_3) \} \]
Intensional/extensional definition

• The set of players of the national football team
  – Extensional definition: enumerating instances
    • \{John, Bob, \ldots\} at 2013
    • \{Dan, Bob, \ldots\} at 2014
  – Intensional definition: conceptual characterization
    • \{x \mid x \text{ is selected as the national team}\}
Intensional relation/function

- A function assigning a relation/function to each world
  - Intensional relation: $\mathcal{W} \rightarrow 2^{D \times \cdots \times D}$
  - Intensional function: $\mathcal{W} \rightarrow (D \times \cdots \times D \rightarrow D)$

Example.

$P: \mathcal{W} \rightarrow 2^{D \times D}$

$P(w_1) = \{(John, Bob)\}$

$P(w_2) = \{(John, Bob), (John, Alice)\}$

$P(w_3) = \{(Alice, Bob)\}$

$P(w_4) = \emptyset$
Interpretation of symbols

• Interpretation $I$ under $\langle W, A, D, R, F \rangle$
  – $I(P) \in R$ for each predicate symbol $P$
  – $I(f) \in F$ for each function symbol $f$
  – $I(c) \in D$ for each constant symbol $c$
  – $I(x) \in D$ for each variable $x$
Semantics of formula

- Let $M = \langle \langle W, A, D, R, F \rangle, I \rangle$.
  - $M, w \models P(t_1, ..., t_n)$ iff $\langle I(w)(t_1), ..., I(w)(t_n) \rangle \in I(P)(w)$
  - $M, w \models \neg \varphi$ iff not $M, w \models \varphi$
  - $\ldots$
  - $M, w \models \forall x \varphi$ iff $M[x \mapsto d], w \models \varphi$ for all $d \in D$
  - $\ldots$
  - $M, w \models \Box \varphi$ iff $M, w' \models \varphi$ for all $(w, w') \in A$
  - $M, w \models \Diamond \varphi$ iff $M, w' \models \varphi$ for some $(w, w') \in A$
Meaning of modal operators

\[ w_1 \models \text{sunny} \]
\[ w_1 \models \neg \Box \text{sunny} \]
\[ w_1 \models \Diamond \text{sunny} \]
\[ w_4 \models \Box \text{sunny} \]
Validity of a formula

• If $M, w \models \varphi$ for all $w \in W$, we say $\varphi$ is valid in $M$ and just write $M \models \varphi$.
Example

Predicate symbol: \{friend\}

\[ P(w_1) = \{(John, Bob)\} \]
\[ P(w_2) = \{(John, Bob), (John, Alice)\} \]
\[ P(w_3) = \{(Alice, Bob)\} \]
\[ P(w_4) = \emptyset \]

Interpretation: \( I = \{\text{friend} \mapsto P\} \)

\[ w_1 \models \text{friend}(John, Bob) \]
\[ w_1 \models \neg \Box \text{friend}(John, Bob) \]
\[ w_4 \models \Box \exists x \text{ friend}(x, Bob) \]
\[ \models \neg \forall x \exists y \text{ friend}(x, y) \]
This talk

• Introduction to logic
  – First-order logic
  – Modal logic

• Information modeling using logic
  – Information vs. knowledge
  – Ontology

• Example ontology
  – Block world
Information vs. knowledge

• Information
  – Individual observations or facts
  – Variable

• Knowledge
  – Systematically organized information
  – Universal
  – Invariable
Example

- 10 is divisible by 2.
- 7 is not divisible by 3.
- ...

Knowledge

Integers are not necessarily divisible by integers.
Knowledge

• Knowledge is a *universal fact* of a certain domain.
  – Information is an instance of knowledge.

• Problem: how we model and formalize knowledge?
This talk

• Introduction to logic
  – First-order logic
  – Modal logic

• Information modeling using logic
  – Information vs. knowledge
  – Ontology

• Example ontology
  – Block world
Ontology

- An *explicit specification* of a *conceptualization*
  - By Gruber 1995

- **Conceptualization** = intended semantic structures (of a certain knowledge domain)
- **Explicit specification** = Vocabulary + logical statements
  - Vocabulary: *symbols* to describe the domain structure
  - Logical statements: *axioms* to restrict the interpretation of symbols
Ontology

• Domain structure = concepts + relationships existing in the domain.

• Ontology describes ... 
  – What are the concepts and relationships existing in the domain?
  – What are the meaning of them?
Language to describe ontology

• Natural language
• Diagrams
• Logic

Informal

Formal
This talk

• Introduction to logic
  – First-order logic
  – Modal logic

• Information modeling using logic
  – Information vs. knowledge
  – Ontology

• Example ontology
  – Block world
Example ontology: block world

Consider to describe the **meaning** of the concepts/relationships “on”, “above”, “clear” and “table” formally.

**Solution?**

\[
\begin{align*}
on &= \{(a, b), (b, c), (d, e)\} \\
above &= \{(a, b), (b, c), (a, c), (d, e)\} \\
clear &= \{a, d\} \\
table &= \{c, e\}
\end{align*}
\]

No! They are just **extensions** of each concept/relationship.
Example ontology: block world

\[ on = \{(a, b), (c, d), (d, e)\} \]
\[ above = \{(a, b), (c, d), (d, e), (c, e)\} \]
\[ clear = \{a, c\} \]
\[ table = \{b, e\} \]

We have a different extension.

The meaning of these concepts/relationships does not change according to the arrangement of blocks.

The meaning should not be defined extensionally but \textbf{intensionally}. 

We have a different extension.
Semantic structure of block world \( \langle W, D, R \rangle \)

- **\( W \):** set of possible worlds
  - Possible arrangement of blocks
- **\( D = \{a, b, c, d, e\} \):**
  - The set of blocks
- **\( R = \{on, above, clear, table\} \):**
  - Set of intensional relations.
Ontology of block world

• We want to **specify** the intensional meaning of concepts/relationships.
  – i.e. what are intended semantic structures of block world.

• We adopt **modal logic** to specify ontology.
Vocabulary of block world

• Predicate symbols: \textit{\underline{on}, \underline{above}, \underline{clear}, \underline{table}}
  – We use overline for each symbol to discriminate from the name of relations in the semantic structure.
Axioms

- \( \neg \forall x(\overline{on}(x, x)) \)

- \( \neg \forall x(\overline{above}(x, x)) \)
  - There is no block which is on/above itself.

- \( \forall x \forall y(\overline{on}(x, y) \rightarrow \overline{above}(x, y)) \)
  - If \( x \) is on \( y \) then \( x \) is above \( y \).

- \( \forall x(\exists y \overline{on}(y, x) \leftrightarrow \overline{clear}(x)) \)
  - If some block \( y \) is on \( x \), then \( x \) is not clear.

- \( \forall x \forall y \forall z(\overline{on}(x, y) \wedge \overline{above}(y, z) \rightarrow \overline{above}(x, z)) \)

- ...

\[
\begin{array}{ccc}
  x & y & x \\
  y & \vdots & \vdots \\
  \vdots & z & z \\
\end{array}
\]
Axioms

- Axioms stipulate the **correct interpretation** of concepts/relationships.
  - Representing universal property (intensional)
  - Not representing individual case (extensional)

- Structures which satisfy the axioms are **the block worlds**.
The following structure is not a block world.

- $W = \{w, \ldots \}$
- $D = \{a, b, c, d, e\}$
- $R = \{\text{on, above, clear, table}\}$
  - $\text{on}(w) = \{(a, b), (b, c)\}$
  - $\text{above}(w) = \{(c, a)\}$
- $I = \{\text{on} \leftrightarrow \text{on}, \text{above} \leftrightarrow \text{above}\}$
Intended semantic structures of block world

- All semantic structures $\mathcal{M} = \langle \langle W, D, R \rangle, I \rangle$ in which the axioms $\Gamma$ are valid.

\[ \{ \mathcal{M} | \mathcal{M} \models \Gamma \} \]
Reasoning about block world

- Theorem. \( \neg \forall a \forall b (\overline{\text{on}}(a, b) \land \overline{\text{on}}(b, a)) \)

- Proof. By **deductive system** of modal logic and axioms.
Additional axiom: rigidity

- Predicate: \textit{block}
  - Means “... is a block.”

- Axiom
  \[ \forall x \left( \text{block}(x) \rightarrow \square \text{block}(x) \right) \]
  - If x is a block, x never becomes but a block.
Other examples of rigidity

- $\forall x(\text{apple}(x) \rightarrow \Box \text{apple}(x))$
- $\forall x(\text{pear}(x) \rightarrow \Box \text{pear}(x))$
- $\neg \forall x(\text{red}(x) \rightarrow \Box \text{red}(x))$
  - “red” is not a rigid property.
Research example

• Formal ontology for business process domain
  – Tasks-Agents-Products (TAP) framework
  – Resource-Agent-Event (REA) framework

• Formal modeling framework for biological systems
  – Gene regulatory network
Conclusion

• First-order logic and modal logic.
• Information vs. knowledge
• Ontology as a knowledge representation.
• Demonstrated that formal logic formally represent knowledge of a certain domain structure.

Formal ontology