Note on "Provability" Predicate※

By

Kenpachiro ŌHASHI

Let S be a formal system containing recursive number theory and Gödel substitution function for a Gödel numbering of S, then $\boldsymbol{B}$ (a) can be constructed which expresses the proposition that the formula with a as its Gödel number is provable in S. And then for $\{A\}$ denoting the Gödel number of the formula A in S, $\boldsymbol{B}$ $(\{A\})$ expresses the proposition that A is provable iu S.

For a suitable "provability" predicate $\boldsymbol{B}$,

$$\vdash A \equiv \vdash \boldsymbol{B}\ (\{A\})$$

may be intuitively true for any formula A, and therefore if "$\rightarrow$" be completely formalized in S, then

$$\vdash A \sim \boldsymbol{B}\ (\{A\})$$

must be derivable in S.

The following remarks apply to many fuuctional calculi,[1] each of which can be variously axiomatized, but for clarity of exposition we shall confine our attention to particular sysems (S). These systems are to have the usual primitive symbols and the following formal axiom schemata, formal rules of inference, and formal recursive number theoretic axioms.

Axiom schema 1 : Any tautol ogous wff (well formed formula)

Axiom schema 2 : $\forall x A(x) \supset A(t)$,

where $A(x)$ is a wff, t is a free variable in A

Axiom schema 3 : $A(t) \supset \exists x A(x)$,

provided that the condition of Axiom schema 2 is satisfeid.

Modus Ponens : applies to wffs A and $A \supset B$, and yields B.

Rule of Generalization : applies to a wff $C \supset A\ (t)$ and yields $C \supset \forall x A\ (x)$, where t and x are not included free in C.

Rule of Existence : applies to a wff $A(t) \supset C$, and yields $\forall x A(x) \supset C$, where t and x are not included free in C.

Axiom schemata 4 :

(1) $A(o)\ \&\ \forall x(A(x) \supset A(x')) \supset A(x)$

(2) $a' = b' \supset a = b$

(3) $\neg\, a' = o$

※ Contribution from the Shimonoseki College of Fisheries, No. 220

(4) $a = b \supset (a = c \supset d = c)$
(5) $a = b \supset a' = b'$
(6) $a + o = a$
(7) $a + b' = (a + b)'$
(8) $a + b = b + a$
(9) $a \cdot o = o$
(10) $a \cdot b' = a \cdot b + a$
(11) $ab = ba$
(12) $a^{o} = 1$
(13) $ab' = ab \cdot a$

Rules of modus ponens, generalization, and existence are called the rule of inference. A formal proof in (S) is a finite column of wffs each of whose lines is a formal axiom or arises from two preceding lines by the Rule of Modus Ponence or arises from a single preceding line by the rule of generalizatuion or of existence.

A wff is called "provable" when there exists a proof the last line of which is the wff $\vdash A$ stands for it that a wff A is provable. Of course, A is not symbol in (S) but of (S).

Since "proof" defined above has finite character, there exists a finite predicate B, such that

(a) $(EY)\ B\ (A,Y) \equiv \vdash A,$

if it is decidable whether a column is a proof of A or not.

If it is not finitely decidable whether a column is a proof of a certain wff, then the theory loses its sense, and then we can assume the above hypothesis, preserving generality.

Y and A will be able to be Gödel numbered, if they have finite character. Let y and a be the Gödel numbers of Y and A respectively, and from (a)

(b) $(Ey)\ R\ (a,y) \equiv \vdash A.$

Moreover, $B\ (A, Y)$ that expresses the proposition Y is a proof of A, and $R\ (a, y)$ can be expressed in (S). Let $\{Y\}$ and $\{A\}$ be the symbols in (S) corresponding to y and a, and from (b)

(c) $(E\ y)$ Bew $(\{A\})$,

which is expressible in (S).

Let $\boldsymbol{B}\ (\{A\})$ stand for (c), and then (a) yields

(d) $\vdash \boldsymbol{B}\ (\{A\}) \equiv \vdash A$

this can be derived into two parts

(e) $\vdash \boldsymbol{B}\ (\{A\}) \rightarrow \vdash A$

(f) $\vdash A \rightarrow \vdash \cdot \boldsymbol{B}\ (\{A\})$

if"$\rightarrow$", which is "implication" in intuitive logic, is completely formalizable in (S), then

(g) $\vdash \boldsymbol{B}\ (\{A\}) \vdash A$

(h) $\vdash A \supset \boldsymbol{B}(\{A\})$

is identically true.

In this paper, it will be proved by the method, which was suggested by *M.* Löb[2] to be available for inconsistency proof, that neither $\vdash A \supset \boldsymbol{B}(\{A\})$ nor $\vdash \boldsymbol{B}(\{A\}) \supset A$ are derivable for any consistent "provability" predicate $\boldsymbol{B}$, and, moreover,

$$\vdash \boldsymbol{B}(\{A \supset B\}) \supset (A \supset B)$$

will be proved not to be derivable.

Terminology and formal and intuitive symbols used here are due to [I].[1]

I. There is no formal consistent "provability" predicate $\boldsymbol{B}$ in any consistent formal system S that satisfies the following conditions:

(1) $\vdash \boldsymbol{B}(\{A\}) \rightarrow \vdash A$

(2) $\vdash A \supset \boldsymbol{B}(\{A\})$

for any formula A.

Proof.

By Gödel substitution function,[3] φ can be constructed which is a formula $\boldsymbol{B}(\{\varphi\}) \supset 1=0$.

1.1 $\vdash \varphi \supset \varphi$

1.2 $\vdash \varphi \supset (\boldsymbol{B}(\{\varphi\}) \supset 1=0$

Assume that

1.3 $\vdash \varphi \supset \boldsymbol{B}(\{\varphi\})$

1.4 $\vdash \varphi \supset 1=0$

1.5 $\vdash \neg \varphi$

1.6 $\vdash \neg (\boldsymbol{B}(\{\varphi\}) \supset 1=0)$

1.7 $\vdash \boldsymbol{B}(\{\varphi\})$ & $\neg(1=0\}$

1.8 $\vdash \varphi$

1.5 and 1.8 contradict the consistency of S.

II. There is no formal consistent "provability" predicate $\boldsymbol{B}$ in any consistent formal system that satisfies the following conditions:

(3) $\vdash A \rightarrow \vdash \boldsymbol{B}(\{A\})$

(4) $\vdash \boldsymbol{B}(\{A\}) \supset A$

Proof.

Let φ be the formula constructed in I.

Suppose that

2.1 $\vdash \boldsymbol{B}(\{\varphi\}) \supset \varphi$

2.2 $\vdash \boldsymbol{B}(\{\varphi\}) \supset (\boldsymbol{B}(\{\varphi\}) \supset 1=0)$

2.3 $\vdash \boldsymbol{B}(\{\varphi\}) \supset 1=0$

2.4 $\vdash \varphi$

2.5 $\vdash \boldsymbol{B}(\{\varphi\})$

2.6 $\vdash 1=0$

this contradicts the consistency of S.

III. For any consistent formal system, there is no "provability" predicate $\boldsymbol{B}$ satisfying

(3) $\vdash A \rightarrow \vdash \boldsymbol{B}(\{A\})$

(5) $\vdash \boldsymbol{B}(\{A \supset B\}) \supset (A \supset B)$

for any formula A and B.

Proof.

Let φ be $B\ (1=1 \supset \varphi) \supset 1=0$.

3.1 $\vdash \boldsymbol{B}(\{A \supset B\}) \supset (A \supset B)$

3.2 $\vdash \boldsymbol{B}(\{1=1 \supset \varphi\}) \supset (1=1 \supset \varphi)$

3.3 $\vdash \boldsymbol{B}(\{1=1 \supset \varphi\}) \supset \varphi$

3.4 $\vdash \boldsymbol{B}(\{1=1 \supset \varphi\}) \supset (\boldsymbol{B}(\{1=1 \supset \varphi\}) \supset 1=0)$

3.5 $\vdash \boldsymbol{B}(\{1=1 \supset \varphi\}) \supset 1=0$

3.6 $\vdash \varphi$

3.7 $\vdash 1=1 \supset \varphi$

3.8 $\vdash \boldsymbol{B}(\{1=1 \supset \varphi\})$

3.9 $\vdash 1=0$

References

1) KLEENE, S. C. [I]. 1952. Introduction into metamathematics. Part III.

2) LÖB, M. H. 1955. Solution of a problem of Leon Henkin. *J. Symb. Logic*, **20** pp. 115—118.

3) HILBERT, D. u. P. Bernays. Grundlagen der Mahematik. II. pp. 279—294.